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Final Paper

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G – Spaces

Let V be a vector space and G a group

Definition: we say that the map φ: G x V→V is an action if

1) φ (e,x)=x $\forall \ x \in V$ where e is the identity of G.
2) φ $(g_1 g_2, x) = \varphi(g_1, \varphi(g_2, x))$ $\forall \ x \in V \ , \forall g_1, g_2 \in G.$

The pair (Y, φ) is called a G-Space. Notation φ (g, x) =gx

Definition: The set {g$\in G$ / φ (g, x) =gx=x} is called subgroup of isotropy at x.

Notation: G_x show that it is a subgroup of G.

The set of {g$\in$G/φ(g,x)=gx=x} is called a subring.

Let

$g_1 g_2 \in Gx_\circ$

$g_1 g_2 \in$

$\varphi(g_1 g_2, x_\circ) = \varphi(g_1, \varphi(g_2, x_\circ)) = \varphi(g_1, x_\circ) = x_\circ$

$Gx_\circ$ is not empty since $g \in Gx_\circ$ e=g^{-1}g

$\varphi(e, x_\circ) = x_\circ \rightarrow \varphi(g^{-1}g, x_\circ) = x_\circ \rightarrow \varphi(g^{-1}, \varphi(g, x_\circ)) = x_\circ \rightarrow \varphi(g^{-1}, x_\circ) = x_\circ \rightarrow g^{-1} \in Gx_\circ$

$\varphi^- : Gx \rightarrow Gy \qquad \varphi^-(g) = hgh^{-1} \qquad$ h(x)=y

$1-$ Well defined $\varphi^-(g) = hgh^{-1}(y) = hg(x) = h(x) = y$

$2 - \varphi^-(g_1) = \varphi^-(g_2)$ we have to show $g_1 = g_2$

$\varphi^-(g_1) = \varphi^-(g_2) \rightarrow hg_1h^{-1} = hg_2h^{-1} \rightarrow h^{-1}hg_1h^{-1} = h^{-1}hg_2h^{-1} \rightarrow g_1h^{-1} = g_2h^{-1} \rightarrow g_1h^{-1}h = g_2h^{-1}h$
$\rightarrow g_1 = g_2$

$\varphi(g_1 g_2, x_\circ) = \varphi(g_1, \varphi(g_2, x_\circ)) = \varphi(g_1, x_\circ) = x_\circ$

$Gx_\circ$ is not empty since $g \in Gx_\circ$ e=g^{-1}g

$\varphi(e, x_\circ) = x_\circ \rightarrow \varphi(g^{-1}g, x_\circ) = x_\circ \rightarrow \varphi(g^{-1}, \varphi(g, x_\circ)) = x_\circ \rightarrow \varphi(g^{-1}, x_\circ) = x_\circ \rightarrow g^{-1} \in Gx_\circ$

Definition: The set { $x \in V$ / g(x)=x, $\forall g \in G$ } is called the set of fixed points under the action of G.

Definition: we say that an action is transitive if $\forall x, y \in V$ $\exists$ g$\in$ G gx=y.

Proposition: suppose φ is transitive. Then $G_x \simeq G_y$ $\forall x, y \in V$ that is any Isotropy subgroups are isomorphic.

Proof: Since φ is transitive then $\exists$ h$\in$ G such that hx=y.

Now define φ^{-}: $G_x \to G_y$ given by φ^{-} (g)=h.g.h^{-1}

Note that φ^{-} is well-defined that is (h.g.h^{-1})(y)=h(g(h^{-1}(y)))=y

Show that φ^{-} is a homomorphism, 1-1 and onto.

$\varphi^{-}: Gx \to Gy \qquad \varphi^{-}(g) = hgh^{-1} \qquad$ h(x)=y

Well defined $\varphi^{-}(g) = hgh^{-1}(y) = hg(x) = h(x) = y$

Show it is 1-1 – $\varphi^{-}(g_1) = \varphi^{-}(g_2)$ we have to show $g_1 = g_2$

$$\varphi^{-}(g_1) = \varphi^{-}(g_2) \to hg_1h^{-1} = hg_2h^{-1} \to h^{-1}hg_1h^{-1} = h^{-1}hg_2h^{-1} \to g_1h^{-1} = g_2h^{-1} \to g_1h^{-1}h = g_2h^{-1}h$$
$$\to g_1 = g_2$$

Show it is homomorphism: $\varphi^{-}(g_1g_2) = \varphi^{-}(g_1)\varphi^{-}(g_2)$

$\varphi^{-}(g_1g_2) = hg_1g_2h^{-1} = hg_1h^{-1}hg_2h^{-1} = (hg_1h^{-1})(hg_2h^{-1}) = \varphi^{-}(g_1)\varphi^{-}(g_2)$

Show it is onto: Let $g^{\sim} \in Gy$

$g^{\sim}(y) = y \to g^{\sim}h(x) = h(x) \to h^{-1}g^{\sim}h = h^{-1}hx \to$ this element is Gx

$\varphi^{-}(g^{\sim}) = x$

Suppose now that V has an inner product <, >. We say that G acts by isometries if

< gx, gy>=<x, y>, $\forall x, y \in V$ $\exists$ g$\in$ G .

Definition: let (V, φ) and (W, ψ) be two G-Spaces and f: V→W.

We say that f is equivariant if

$f \circ g = g \circ f$ $\forall$g$\in$ G.

That is if $f(gx) = g(fx)$ $\forall$x$\in$ V and g$\in$ G.

A map f: V→V is said to be invariant if $f \circ g = f$ $\forall$g$\in$ G that is f (gx) =f(x)$\forall$x$\in$ V and g$\in$ G .

Definition: Let S $\subset$ V. We say that S is invariant by the action of G if G(s) $\subset$ S. By G(s) we mean G(s) = {gs/ $\forall s \in S$, $g \in G$ }

Fixing s $\in$ S, G(s) = {gs/ $\forall g \in G$ } is called orbit of G at s.

Note that if φ^{-} is transitive, then the orbit of point is the entire S.

Now let us consider an invariant set S, the isotropy subgroup of $x_\circ \in S$ and let us assume that φ is transitive on S.

Define $\varphi^{-} : G / G_{x_\circ} \to S$ Let H denote $G_{x_\circ}$, $gH \mapsto \varphi(g, x_\circ) = g x_\circ$

1- φ^{-} is well defined. In fact if

$g_1 H = g_2 H$, then $g_2^{-1} g_1 \in H$ and hence $g_2^{-1} g_1 (x_\circ) = x_\circ \Rightarrow g_1 x_\circ = g_2 x_\circ$

2- φ^{-} is onto. In fact, since φ is transitive, given x, $\forall g \in G$ such that $g x_\circ = x$ therefore $\varphi^{-}(gH) = x$.

3- φ^{-} is 1-1, since for $g_1 x_\circ = g_2 x_\circ \Rightarrow g_2^{-1} g_1 (x_\circ) = x_\circ \Rightarrow g_2^{-1} g_1 \in H \Rightarrow g_1 H = g_2 H$

Example: Consider the sphere S^{n-1}={ $x \in \mathbb{R}^n / x_1^2 + x_2^2 + ... + x_n^2 = 1$ }

Let G=SO (n)={ $A \in M_n(\mathbb{R}) / A.A^t = I \ \det A = 1$ }

Show that G is a group.

$Let \ A_1, A_2 \in G \to \det(A_1) = 1, \det(A_2) = 1$

We have to show $A_1 A_2 \in G$

$\det(A_1 A_2) = \det(A_1)\det(A_2) = 1x1 = 1 \to A_1 A_2 \in G$

$\det I_n = 1 \ \ 1 \in G$

if $A \in G$ We have to show $A^{-1} \in G$

Since det(A)=1 so not zero it is invertible and A^{-1} *exists*

$$\det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{1} = 1 \to A^{-1} \in G$$

A $\leftrightarrow$ T, an orthogonal transformation: T: $\mathbb{R}^n \to \mathbb{R}^n$, That is <T(v), T(w)> =<v,w> $\forall v, w, \in \mathbb{R}^n$ that preserves orientation.

Properties of Orthogonal Transformations:

1)- T Maps orthonormal bases onto orthonormal bases.

2)- Let { $v_1, v_2 .....v_n$ } be and orthonormal basis of $\mathbb{R}^n$. Write each v_i= $\left(x^i{}_1.....x^i{}_n\right)$ and the change of coordinates matrix of T is $A = \begin{bmatrix} x_1^1 & \ldots & x_n^1 \\ \ldots & \ldots & \ldots \\ x_1^n & \ldots & x_n^n \end{bmatrix}$ and its transpose is

$A^t = \begin{bmatrix} x_1^1 & \ldots & x_1^n \\ \ldots & \ldots & \ldots \\ x_n^1 & \ldots & x_n^n \end{bmatrix}$ x note that A A^t=A^t A=I

Define now the action φ: SO(n) x S^{n-1}→ S^{n-1} by φ(A, X)=AX ∈ → S^{n-1}

Show that φ is an action.

We show now that φ is transitive that is , given v_1, u_1∈ S^{n-1}, there exists a matrix C ∈ SO(n) such that C.v_1=u_1

Given u_1, we consider a basis β={ $v_1, v_2 .....v_n$ }. As we before we write u_i=$\left(x^i{}_1.....x^i{}_n\right)$} and get the orthogonal matrix A= [x_i^j].Similarly , given v_1 we consider an orthogonal basis β'={ $v_1, v_2 .....v_n$ } and the matrix B=[y_i^j] where v_i= $\left(y^i{}_1.....y^i{}_n\right)$.

Note that A. $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ = A.e_1=u_1 and B.e_1=v_1 therefore e_1=B^{-1} v_1 → A.e_1=A B^{-1} v_1 =u_1

Conclusion: det(AB^{-1})=1?? We take C= AB^{-1} ∈ SO(n).

Now we find the isotropy subgroup of e_n=(0,0,..0,1), G_{e_n} .

Let H={A ∈ SO(n)/A=$\begin{pmatrix} & & 0 \\ & A^{\sim} & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$, $A^{\sim} \in SO(n-1)$ }

We know that H ⊂ G_{e_n} . In order to prove that G_{e_n} ⊂H, we consider

A=(a_{ij}) A. $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$ = $\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{bmatrix}$ = $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$ → a_{in}=0 for i<n and $a_{n\,n}$=1

We also have that A.A^t=I→ $\sum_{i=1}^{n} a_{ni}^2 = 1$ Since $a_{n\,n}$=1→ a_{ni}=0 for all i<n.

Therefore A= $\begin{pmatrix} & & 0 \\ & \tilde{A} & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$ Since det A= det $\tilde{A}$, we get that det $\tilde{A}$ =1

Note also that A.A^t = $\begin{pmatrix} & & 0 \\ & \tilde{A}\,\tilde{A} & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$ =I $\longrightarrow$ $\tilde{A}.\tilde{A}^t$ is the (n-1) x (n-1) identity matrix.

So $\tilde{A} \in$ SO(n). We can write $S^{n-1} = \frac{SO(n)}{SO(n-1)}$